The wavefunctions for the free-fermion part of the spectrum of the $SU_q(N)$ quantum spin models

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Abstract
We conjecture that the free-fermion part of the eigenspectrum observed recently for the $SU_q(N)$ Perk–Schultz spin chain Hamiltonian in a finite lattice with $q = \exp(i\pi(N-1)/N)$ is a consequence of the existence of a special simple eigenvalue for the transfer matrix of the auxiliary inhomogeneous $SU_q(N-1)$ vertex model which appears in the nested Bethe ansatz approach. We prove that this conjecture is valid for the case of the $SU_q(3)$ spin chain with periodic boundary condition. In this case we obtain a formula for the components of the eigenvector of the auxiliary inhomogeneous 6-vertex model ($q = \exp(2i\pi/3)$), which permits us to find one by one all components of this eigenvector and consequently to find the eigenvectors of the free-fermion part of the eigenspectrum of the $SU_q(3)$ spin chain. Similarly, as in the known case of the $SU_q(2)$ case at $q = \exp(i2\pi/3)$ our numerical and analytical studies induce some conjectures for special rates of correlation functions.

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1. Introduction

It was found very recently that part of the eigenspectrum of some quantum spin models is given by a sum of free-fermion quasienergies [1, 2]. In particular, the ground state energy of the $SU_q(N)$ invariant Perk–Schultz Hamiltonian

$$H_q = \sum_{j=1}^{L-1} H_{j,j+1} \tag{1}$$

where

$$H_{i,j} = -\sum_{a=0}^{N-1}\sum_{b=a+1}^{N-1} \left(E_i^{ab} E_j^{ba} + E_i^{ba} E_j^{ab} - q E_i^{aa} E_j^{bb} - 1/q E_i^{bb} E_j^{aa} \right)$$

and E^{ab} are $N \times N$ matrices with elements $(E^{ab})_{cd} = \delta_c^a \delta_d^b$, is given by

$$E_0 = 1 + 2(1 - L + n) \cos\left(\frac{\pi}{N}\right) - \frac{\sin \pi (2n + 1)/2L}{\sin \pi/2L} \tag{2}$$

for the special value of the deformation parameter

$$q = \exp\left(\frac{i\pi (N - 1)}{N}\right). \tag{3}$$

The lattice size defining the quantum chain is L and the parameter n in (2) is given by the integer part of L/N.

Besides the ground state energy (2) for the $SU_q(N)$ model the $SU_q(N-1)$ quantum chain with the same special value of the anisotropy (3) has a ground state energy given by a quite simple formula

$$\tilde{E}_0 = -2(L - 1) \cos\left(\frac{\pi}{N}\right). \tag{4}$$

It is also important to mention that at the special anisotropy (3) the $SU_q(N)$ model exhibits many eigenenergies that can be expressed as the sum of free-fermion quasienergies. In contrast, at the anisotropy (3) the $SU_q(N-1)$ model exhibits a single energy level on this class, namely the ground state whose energy is given in (4). Actually this last eigenstate can be considered as a special state of the $SU_q(N)$ model in the sector where only $(N-1)$ classes of particles are present (see conjecture 3 of [2]). We intend to show in this paper that the corresponding wavefunction possesses nice combinatorial properties and its components play a very important role in the nested Bethe ansatz approach being used for the construction of all other free-fermion eigenstates. This comes from the fact that using the nested Bethe ansatz (NBA) method for the $SU_q(N)$ invariant Hamiltonian (1) we obtain (see, e.g., [3]) an auxiliary transfer matrix of the inhomogeneous $SU_q(N-1)$ invariant vertex model. We conjecture that this matrix has a unique factorizable eigenvalue that reduces some of the NBA equations to a very simple form leading to the free-fermion-like structure. As a result we obtain the free-fermion part of the eigenspectrum of the Hamiltonian (1). All related eigenvectors can be found from the knowledge of this unique eigenvector of the transfer matrix of the inhomogeneous $SU_q(N-1)$ vertex model. In the homogeneous case the transfer matrix commutes with the $SU_q(N-1)$ invariant Hamiltonian and this special eigenvector reduces to the ground state eigenvector with energy (4).

As far as a periodic boundary case is concerned, we have found the free-fermion spectrum only for the $SU(3)$ Perk–Schultz model with $q = \exp(2i\pi/3)$ (we do not consider here the free-fermion point of the $SU(2)$ model, or equivalently the standard XY spin chain). We show below that the free-fermion part of the eigenspectrum of the $SU(3)$ model with periodic boundary condition can be explained by the existence of a unique factorizable eigenvalue for the transfer matrix of the inhomogeneous 6-vertex model with $q = \exp(2i\pi/3)$. We prove the existence of this eigenvalue using different methods described in [2, 4]. Comparison of these approaches sheds a new light on recently discovered combinatorial properties of the ground state wavefunction for the odd length XXZ spin model with $\Delta = -1/2$ [5–7] (see also [8] for similar results for the XYZ spin chain).

2. The $SU_q(N)$ invariant model

The Hamiltonian (1) defines a special dynamics of localized quantum $SU(N)$ spins on a one-dimensional lattice [9, 10]. Identifying the possible values of the localized spins as distinct classes of particles the Hamiltonian (1) describes the fluctuations of a system containing N

classes of particles $(0, 1, \ldots, N-1)$, with a restriction of single occupancy at each lattice site. The number of particles belonging to each class is conserved separately. Consequently we can separate the Hilbert space into block disjoint sectors labelled by $(n_0, n_1, \ldots, n_{N-1})$, where $n_i = 0, 1, \ldots, L$ is the number of particles of class i $(i = 0, 1, \ldots, N-1)$. The Hamiltonian has an S_N symmetry due to its invariance under the permutation of particles species, which implies that all the energies can be obtained from the sectors $(n_0, n_1, \ldots, n_{N-1})$, where $n_0 \leqslant n_1 \leqslant \cdots \leqslant n_{N-1}$ and $n_0 + n_1 + \cdots + n_{N-1} = L$. Moreover, the quantum $SU(N)_q$ symmetry of H implies that all energies in the sector $(n'_0, n'_1, \ldots, n'_{N-1})$ with $n'_0 \leqslant n'_1 \leqslant \cdots \leqslant n'_{N-1}$ are degenerate with the energies belonging to the sectors $(n_0, n_1, \ldots, n_{N-1})$ with $n_0 \leqslant n_1 \leqslant \cdots \leqslant n_{N-1}$, if $n'_0 \leqslant n_0$ and $n'_0 + n'_1 \leqslant n_0 + n_1$ and so on up to $n'_0 + n'_1 + \cdots + n'_{N-2} \leqslant n_0 + n_1 + \cdots + n_{N-2}$.

The nested Bethe ansatz equations (NBAE) for the $SU_q(N)$ Perk–Schultz model (1) are given by (see, e.g., [3, 11–13])

$$\prod_{j=1, j\neq i}^{p_k} F\left(u_i^{(k)}, u_j^{(k)}\right) = \prod_{j=1}^{p_{k-1}} f\left(u_i^{(k)}, u_j^{(k-1)}\right) \prod_{j=1}^{p_{k+1}} f\left(u_i^{(k)}, u_j^{(k+1)}\right) \tag{5}$$

where $k = 0, 1, \ldots, N-2$ and $i = 1, 2, \ldots, p_k$. The integer parameters p_k depend on the particle occupation numbers $\{n_i\}$:

$$p_k = \sum_{i=0}^{k} n_i \qquad k = 0, 1, \ldots, N-2 \qquad p_{-1} = 0 \qquad p_{N-1} = L \tag{6}$$

and the functions $F(x, y)$ and $f(x, y)$ are defined by

$$F(x, y) = \frac{\cos(2y) - \cos(2x - 2\eta)}{\cos(2y) - \cos(2x + 2\eta)} \qquad f(x, y) = \frac{\cos(2y) - \cos(2x - \eta)}{\cos(2y) - \cos(2x + \eta)}. \tag{7}$$

In the NBAE (5) we have variables of different classes. The number of variables $u_i^{(k)}$ of class k is equal to p_k and the variables $u_i^{(N-1)} = 0$ $(i = 1, \ldots, L)$. The whole system of NBAE consists of subsets of equations labelled by k and containing precisely p_k equations $(k = 0, 1, \ldots, N-2)$.

The eigenenergies of the Hamiltonian (1) in the sector $(n_0, n_1, \ldots, n_{N-1})$ are given by

$$E = -\sum_{j=1}^{p} \left(-q - \frac{1}{q} + \frac{\sin(u_j - \eta/2)}{\sin(u_j + \eta/2)} + \frac{\sin(u_j + \eta/2)}{\sin(u_j - \eta/2)} \right) \tag{8}$$

where to simplify the notation $p \equiv p_{N-2}$ and $u_j \equiv u_j^{(N-2)}$, $j = 1, 2, \ldots, p$.

All the solutions of NBAE (5) described in this and our previous paper [2] satisfy the additional 'free-fermion' conditions (FFC)

$$f^L(u_i, 0) = 1 \qquad i = 1, \ldots, p. \tag{9}$$

Consequently from (8) and (9) the corresponding eigenenergies of the Hamiltonian (1) are given by

$$E = -2\sum_{j=1}^{p_{N-2}} \left(-\cos\eta + \cos\frac{\pi k_j}{L} \right) \qquad 1 \leqslant k_j \leqslant L-1 \tag{10}$$

where $\{k_j\}$ is any set of distinct integers in the range $1 \leqslant k_j \leqslant L-1$.

In the derivation of the NBAE (5) it is necessary to find the eigenvalues of an auxiliary matrix that corresponds to the transfer matrix of an inhomogeneous $SU_q(N-1)$ invariant

vertex model on the square lattice of width p in the horizontal direction. These eigenvalues enter the NBAE (see equation (51) from [3])[3]:

$$\Lambda_{\text{aux}}^{(N-1)}(u_i) = f^{-L}(u_i, 0) \prod_{j=1, j\neq i}^{p} F(u_i, u_j) \qquad i = 1, \ldots, p. \tag{11}$$

We see that the FFC (9) could be explained if there exists a special eigenvalue for this transfer matrix:

$$\Lambda_{\text{aux}}^{(N-1)}(u) = F^{-1}(u, u) \prod_{j=1}^{p} F(u, u_j). \tag{12}$$

This observation can be formulated in the following conjecture.

Conjecture 1. *Let $q = \exp \frac{i\pi(N-1)}{N}$ and consider the inhomogeneous $SU_q(N-1)$ invariant vertex model on the square lattice with p columns, where $p = (N-1)k + r$, and $0 \leqslant r \leqslant N-2$. The inhomogeneity of the model in the horizontal direction is fixed by the vertical rapidities u_j, $j = 1, \ldots, p$. The row-to-row transfer matrix, depending on the spectral parameter u (horizontal rapidity), has a special factorizable eigenvalue given by formula (12). The corresponding eigenvector belongs to the sector $S = \{n_0, n_1, \ldots, n_{N-2}\}$ where $n_i = k$, $i = 0, \ldots, r-1$, and $n_i = k+1$ for $i = r, \ldots, N-2$.*

3. The $SU(3)$ Perk–Schultz model with periodic boundary

The $SU(3)$ Perk–Schultz model [10] is the anisotropic version of the $SU(3)$ Sutherland model [9], with the Hamiltonian, in a periodic L-site chain, given by

$$H_q = \sum_{j=1}^{L} H_{j,j+1} \qquad (H_{L,L+1} \equiv H_{L,1})$$

$$H_{i,j} = -\sum_{a=0}^{1} \sum_{b=a+1}^{2} \left(E_i^{ab} E_j^{ba} + E_i^{ba} E_j^{ab} - q E_i^{aa} E_j^{bb} - 1/q E_i^{bb} E_j^{aa} \right). \tag{13}$$

In our previous paper (see [1] for details) it was shown for the periodic model that:

The Hamiltonian (13) with L sites at $q = \exp(2i\pi/3)$ has eigenvectors (not all of them) with energy and momentum given by

$$E_I = -\sum_{j \in I} \left(1 + 2 \cos \frac{2\pi j}{L} \right) \tag{14}$$

$$P_I = \frac{2\pi}{L} \sum_{j \in I} j \tag{15}$$

with I being a subset of $\mathcal{I}$ unequal elements of the set $\{1, 2, \ldots, L\}$. The number $\mathcal{I}$ has to be odd $\mathcal{I} = 2k+1$ and the sector of appearance of the above levels is $S_k \equiv (k, k+1, L-2k-1), 0 \leqslant k \leqslant (L-1)/2$.

The corresponding solutions of the NBAE were described in [2] and we intend now to present a procedure that allows us to calculate the related wavefunctions. This procedure is based on the coordinate Bethe ansatz method and we follow here [14].

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[3] All details related to the construction of the transfer matrix can be found in [3].

Due to the conservation of particles the total numbers of particles n_0, n_1 and $n_2 = L - n_0 - n_1$ on classes 0, 1 and 2 are good quantum numbers, and consequently we can split the associated Hilbert space into block disjoint sectors labelled by the numbers n_0 and n_1. We consider the eigenvalue equation

$$H|n_0, n_1\rangle = E|n_0, n_1\rangle \tag{16}$$

where

$$|n_0, n_1\rangle = \sum_{\{Q\}} \sum_{\{x\}} f(x_1, Q_1; \ldots; x_n, Q_n)|x_1, Q_1; \ldots; x_n, Q_n\rangle \tag{17}$$

and $n = n_0 + n_1$. Here $|x_1, Q_1; \ldots; x_n, Q_n\rangle$ means the configuration where a particle of class Q_i ($Q_i = 0, 1$) is at position x_i ($x_i = 1, \ldots, L$). The summation $\{Q\} = \{Q_1, \ldots, Q_n\}$ extends over all $(0, 1)$ sequences in which n_0 terms are 0 and n_1 terms are 1. The summation $\{x\} = \{x_1, \ldots, x_n\}$ extends to n increasing positive integers satisfying $1 \leqslant x_1 < \cdots < x_n \leqslant L$. Before getting the results for general values of n let us consider initially the simple cases where we have 1 or 2 particles.

Case 1: $n = 1$. For one particle on the chain (class 0 or 1), as a consequence of the translational invariance of (13) it is simple to verify directly that the eigenfunctions are the momentum-k eigenfunctions

$$|1, 0\rangle = \sum_{x=1}^{L} f(x, 0)|x, 0\rangle \qquad \text{or} \qquad |0, 1\rangle = \sum_{x=1}^{L} f(x, 1)|x, 1\rangle \tag{18}$$

with

$$f(x, 0) = f(x, 1) = e^{ikx} \qquad k = \frac{2\pi l}{L} \qquad l = 0, 1, \ldots, L - 1 \tag{19}$$

and energy given by

$$E = e(k) \equiv -e^{ik} - e^{-ik} + q + 1/q. \tag{20}$$

Case 2: $n = 2$. For two particles of classes Q_1 and Q_2 ($Q_1, Q_2 = 0, 1$) on the lattice, the eigenvalue equation (16) gives us two distinct relations depending on the relative location of the particles. The first relation comes from the amplitudes where a particle of class Q_1 is at position x_1 and a particle Q_2 is at position x_2, where $x_2 > x_1 + 1$. We obtain in this case the relation

$$\begin{aligned}
Ef(x_1, Q_1; x_2, Q_2) = &-f(x_1 - 1, Q_1; x_2, Q_2) - f(x_1, Q_1; x_2 + 1, Q_2) \\
&- f(x_1 + 1, Q_1; x_2, Q_2) - f(x_1, Q_1; x_2 - 1, Q_2) \\
&+ 2(q + 1/q) f(x_1, Q_1; x_2, Q_2).
\end{aligned} \tag{21}$$

This last equation can be solved promptly by the ansatz

$$f(x_1, Q_1; x_2, Q_2) = e^{ik_1 x_1} e^{ik_2 x_2} \tag{22}$$

with energy

$$E = e(k_1) + e(k_2). \tag{23}$$

Since this relation is symmetric under the interchange of k_1 and k_2, we can write a more general solution of (21) as

$$\begin{aligned}
f(x_1, Q_1; x_2, Q_2) &= \sum_P A_{P_1, P_2}^{Q_1, Q_2} e^{i(k_{P_1} x_1 + k_{P_2} x_2)} \\
&= A_{1,2}^{Q_1, Q_2} e^{i(k_1 x_1 + k_2 x_2)} + A_{2,1}^{Q_1, Q_2} e^{i(k_2 x_1 + k_1 x_2)}
\end{aligned} \tag{24}$$

with the same energy as in (23). In the last equation the summation is over the permutations $P = P_1, P_2$ of $(1, 2)$. The second relation comes from the amplitude where $x_2 = x_1 + 1$ (matching condition). In this case instead of (21) we have

$$
\begin{aligned}
Ef(x_1, Q_1; x_1+1, Q_2) &= -f(x_1-1, Q_1; x_1+1, Q_2) - f(x_1, Q_1; x_1+2, Q_2) \\
&\quad - f(x_1, Q_2; x_1+1, Q_1) + (2q+1/q)f(x_1, Q_1; x_1+1, Q_2) \qquad Q_1 < Q_2 \\
Ef(x_1, Q_1; x_1+1, Q_2) &= -f(x_1-1, Q_1; x_1+1, Q_2) - f(x_1, Q_1; x_1+2, Q_2) \\
&\quad + (q+1/q)f(x_1, Q_1; x_1+1, Q_2) \qquad Q_1 = Q_2 \\
Ef(x_1, Q_1; x_1+1, Q_2) &= -f(x_1-1, Q_1; x_1+1, Q_2) - f(x_1, Q_1; x_1+2, Q_2) \\
&\quad - f(x_1, Q_2; x_1+1, Q_1) + (q+2/q)f(x_1, Q_1; x_1+1, Q_2) \qquad Q_1 > Q_2.
\end{aligned} \tag{25}
$$

Since the ansatz (24) with (23) is also a solution of (21) with $x_2 = x_1 + 1$, we obtain from (25)

$$
\begin{aligned}
f(x_1, Q_1; x_1, Q_2) &+ f(x_1+1, Q_1; x_1+1, Q_2) \\
&= (1/q)f(x_1, Q_1; x_1+1, Q_2) + f(x_1, Q_2; x_1+1, Q_1) \qquad Q_1 < Q_2 \\
f(x_1, Q_1; x_1, Q_2) &+ f(x_1+1, Q_1; x_1+1, Q_2) \\
&= (q+1/q)f(x_1, Q_1; x_1+1, Q_2) \qquad Q_1 = Q_2 \\
f(x_1, Q_1; x_1, Q_2) &+ f(x_1+1, Q_1; x_1+1, Q_2) \\
&= qf(x_1, Q_1; x_1+1, Q_2) + f(x_1, Q_2; x_1+1, Q_1) \qquad Q_1 > Q_2.
\end{aligned} \tag{26}
$$

If we now substitute the ansatz (24) into these equations the constants $A_{1,2}^{Q_1,Q_2}$ and $A_{2,1}^{Q_1,Q_2}$, initially arbitrary, should now satisfy

$$
\begin{aligned}
&\sum_P \left\{ \left(\sigma_{P_1,P_2} + q\,e^{ik_{P_2}} \right) A_{P_1,P_2}^{0,1} - e^{ik_{P_2}} A_{P_1,P_2}^{1,0} \right\} = 0 \\
&\sum_P \sigma_{P_1,P_2} A_{P_1,P_2}^{Q,Q} = 0 \qquad Q = 1, 2 \\
&\sum_P \left\{ \left(\sigma_{P_1,P_2} + q^{-1}\,e^{ik_{P_2}} \right) A_{P_1,P_2}^{1,0} - e^{ik_{P_2}} A_{P_1,P_2}^{0,1} \right\} = 0
\end{aligned} \tag{27}
$$

where

$$
\sigma_{P_1,P_2} = 1 + e^{ik_{P_1}+ik_{P_2}} - (q+q^{-1})\,e^{ik_{P_2}}. \tag{28}
$$

The system (27) consists of three equations. The second equation can be easily rewritten as

$$
A_{P_1,P_2}^{Q,Q} = -\frac{\sigma_{P_2,P_1}}{\sigma_{P_1,P_2}} A_{P_2,P_1}^{Q,Q} \tag{29}
$$

and combining the first and the third equations we obtain

$$
A_{P_1,P_2}^{0,1} = -\frac{(1 - q\,e^{ik_{P_2}})(1 - q^{-1}e^{ik_{P_1}})}{\sigma_{P_1,P_2}} A_{P_2,P_1}^{0,1} + \frac{(e^{ik_{P_1}} - e^{ik_{P_2}})}{\sigma_{P_1,P_2}} A_{P_2,P_1}^{1,0} \tag{30}
$$

$$
A_{P_1,P_2}^{1,0} = +\frac{(e^{ik_{P_1}} - e^{ik_{P_2}})}{\sigma_{P_1,P_2}} A_{P_2,P_1}^{0,1} - \frac{(1 - q\,e^{ik_{P_1}})(1 - q^{-1}\,e^{ik_{P_2}})}{\sigma_{P_1,P_2}} A_{P_2,P_1}^{1,0}. \tag{31}
$$

Equations (29)–(31) can be written in a compact form

$$
A_{P_1,P_2}^{Q_1,Q_2} = -\sum_{Q_1',Q_2'=0}^{1} S_{Q_2',Q_1'}^{Q_1,Q_2}\left(k_{P_1}, k_{P_2}\right) A_{P_2,P_1}^{Q_1',Q_2'} \qquad (Q_1, Q_2 = 0, 1) \tag{32}
$$

where we have introduced the S matrix

$$S_{0,0}^{0,0}(k_1, k_2) = S_{1,1}^{1,1}(k_1, k_2) = \frac{\sigma_{2,1}}{\sigma_{1,2}}$$

$$S_{1,0}^{0,1}(k_1, k_2) = \frac{(1 - q\, e^{ik_2})(1 - q^{-1}\, e^{ik_1})}{\sigma_{1,2}}$$

$$S_{0,1}^{1,0}(k_1, k_2) = \frac{(1 - q\, e^{ik_1})(1 - q^{-1}\, e^{ik_2})}{\sigma_{1,2}} \tag{33}$$

$$S_{0,1}^{0,1}(k_1, k_2) = S_{1,0}^{1,0}(k_1, k_2) = \frac{(e^{ik_2} - e^{ik_1})}{\sigma_{1,2}}.$$

Equations (29)–(31) or (32) do not fix the 'wave numbers' k_1, k_2. In general, these numbers are fixed due to the cyclic boundary conditions:

$$f(x_1, Q_1; x_2, Q_2) = f(x_2, Q_2; x_1 + L, Q_1) \tag{34}$$

which from (24) give us the relations

$$A_{P_1,P_2}^{Q_1,Q_2} = A_{P_2,P_1}^{Q_2,Q_1}\, e^{ik_{P_1} L}. \tag{35}$$

This last equation, when solved by exploiting (32) and (33), gives us the possible values of k_1 and k_2, and from (23) the eigenenergies in the sectors containing two particles. Instead of solving these equations for the particular case $n = 2$ let us now consider the case of general n.

Case 3: *general n*. The above calculation can be generalized for arbitrary values of n_0 and n_1 of the particles of classes 0 and 1, respectively ($n_0 + n_1 = n$). The ansatz for the wavefunction (17) becomes

$$f(x_1, Q_1; \ldots; x_n, Q_n) = \sum_P A_{P_1,\ldots,P_n}^{Q_1,\ldots,Q_n}\, e^{i(k_{P_1} x_1 + \cdots + k_{P_n} x_n)} \tag{36}$$

where the sum extends over all permutations P of the integers $1, 2, \ldots, n$. It is simple to see that the relations coming from the eigenvalue equation (16) for the components $|x_1, Q_1; \ldots; x_n, Q_n\rangle$ where $x_{i+1} - x_i > 1$ for $i = 1, 2, \ldots, n$ are satisfied by the ansatz (36) with energy

$$E = \sum_{j=1}^{n} e(k_j). \tag{37}$$

On the other hand, if a pair of particles belonging to classes Q_i, Q_{i+1} is located at positions x_i, x_{i+1}, where $x_{i+1} = x_i + 1$, equation (16) with the ansatz (36) and relation (37) give us the generalization of relation (32), namely,

$$A_{\ldots,P_i,P_{i+1},\ldots}^{\ldots,Q_i,Q_{i+1},\ldots} = -\sum_{Q_1',Q_2'=0}^{1} S_{Q_1',Q_2'}^{Q_i,Q_{i+1}}(k_{P_i}, k_{P_{i+1}}) A_{\ldots,P_{i+1},P_i,\ldots}^{\ldots,Q_2',Q_1',\ldots} \qquad Q_i, Q_{i+1} = 0, 1 \tag{38}$$

with S given by (33). Inserting the ansatz (36) in the boundary condition

$$f(x_1, Q_1; \ldots; x_n, Q_n) = f(x_2, Q_2; \ldots; x_n, Q_n; x_1 + L, Q_1) \tag{39}$$

we obtain the additional relation

$$A_{P_1,\ldots,P_n}^{Q_1,\ldots,Q_n} = e^{ik_{P_1} L} A_{P_2,\ldots,P_n,P_1}^{Q_2,\ldots,Q_n,Q_1} \tag{40}$$

which together with (38) should give us the eigenenergies.

Successive applications of (38) give us in general distinct relations between the amplitudes. They are consistent because as we will see below the S matrix (33) coincides with the famous 6-vertex R matrix and satisfies the Yang–Baxter equation. Hence we may use these relations

to obtain the eigenenergies of the Hamiltonian (13). Applying relation (38) n times we obtain from (40) a relation connecting the amplitudes with the same momenta, namely,

$$A_{P_1,\ldots,P_n}^{Q_1,\ldots,Q_n} = e^{ik_{P_1}L} A_{P_2,\ldots,P_n,P_1}^{Q_2,\ldots,Q_n,Q_1} = (-1)^{n-1} e^{ik_{P_1}L}$$

$$\times \sum_{Q_1',\ldots,Q_n'} \sum_{Q_1'',\ldots,Q_n''} S_{Q_1',Q_1''}^{Q_1,Q_2''}(k_{P_1},k_{P_1}) S_{Q_2',Q_2''}^{Q_2,Q_3''}(k_{P_2},k_{P_1}) \cdots \qquad (41)$$

$$S_{Q_{n-1}',Q_{n-1}''}^{Q_{n-1},Q_n''}(k_{P_{n-1}},k_{P_1}) S_{Q_n',Q_n''}^{Q_n,Q_1''}(k_{P_n},k_{P_1}) A_{P_1,\ldots,P_n}^{Q_1',\ldots,Q_n'}$$

where we have introduced the harmless extra sum

$$1 = \sum_{Q_1'',Q_2''=0}^{1} \delta_{Q_2'',Q_1'}\delta_{Q_1'',Q_1} = \sum_{Q_1'',Q_2''=0}^{1} S_{Q_1',Q_1''}^{Q_1,Q_2''}(k_{P_1},k_{P_1}). \qquad (42)$$

In order to fix the values of $\{k_j\}$ we should solve (41), i.e., we should find the eigenvalues $\Lambda(k)$ of the matrix

$$\mathcal{T}(k)_{\{Q'\}}^{\{Q\}} = \sum_{Q_1'',\ldots,Q_n''=0}^{1} \left\{ \left(\prod_{l=1}^{n-1} S_{Q_l',Q_l''}^{Q_l,Q_{l+1}''}(k_{P_l},k) \right) S_{Q_n',Q_n''}^{Q_n,Q_1''}(k_{P_n},k) \right\} \qquad (43)$$

and the Bethe ansatz equations which fix the set $\{k_l\}$ will be given from (41) by

$$e^{-ik_jL} = (-1)^{n-1}\Lambda(k_j) \qquad j = 1,\ldots,n. \qquad (44)$$

We identify $\mathcal{T}(k)$ as the transfer matrix of an inhomogeneous 6-vertex model, on a periodic lattice, with Boltzmann weights $S_{Q_1',Q_2'}^{Q_1,Q_2}(k_{P_l},k)$ ($l = 1,\ldots,n$). Consequently, in order to obtain the eigenenergies of the quantum chain (13) we should diagonalize the above transfer matrix $T(k)$.

It is convenient to introduce the variables $\{u_j\}$ as in the NBAE of section 2:

$$e^{-ik} = \frac{\sin(u - \pi/3)}{\sin(u + \pi/3)}. \qquad (45)$$

In terms of these variables the S matrix (33) has a difference form

$$S_{0,0}^{0,0}(u_1 - u_2) = S_{1,1}^{1,1}(u_1 - u_2) = \frac{\sin(u_1 - u_2 + \pi/3)}{\sin(u_2 - u_1 + \pi/3)}$$

$$S_{1,0}^{0,1}(u_1 - u_2) = \frac{\sqrt{3}\, e^{i(u_1-u_2)}}{2\sin(u_2 - u_1 + \pi/3)}$$

$$S_{0,1}^{1,0}(u_1 - u_2) = \frac{\sqrt{3}\, e^{i(u_2-u_1)}}{2\sin(u_2 - u_1 + \pi/3)} \qquad (46)$$

$$S_{0,1}^{0,1}(u_1 - u_2) = S_{1,0}^{1,0}(u_1 - u_2) = \frac{\sin(u_2 - u_1)}{\sin(u_2 - u_1 + \pi/3)}.$$

where we should recall that we are considering $q = \exp(2i\pi/3)$.

Introducing for convenience the new amplitudes

$$A_{P_1,\ldots,P_n}^{Q_1,\ldots,Q_n} = \exp\left(i\sum_{j=1}^{n} \delta_{Q_j,0}\, u_{P_j} \right) \tilde{A}_{P_1,\ldots,P_n}^{Q_1,\ldots,Q_n} \qquad (47)$$

we obtain instead of (38) a similar relation with $\tilde{A}$ instead of A and with the $\tilde{S}$ matrix of the symmetric 6-vertex model, whose non-zero components are given by

$$\tilde{S}_{0,0}^{0,0}(u) = \tilde{S}_{1,1}^{1,1}(u) = \rho\,\sin(\pi/3 - u)$$

$$\tilde{S}_{0,1}^{0,1}(u) = \tilde{S}_{1,0}^{1,0}(u) = \rho\,\sin u \qquad (48)$$

$$\tilde{S}_{1,0}^{0,1}(u) = \tilde{S}_{0,1}^{1,0}(u) = \rho\,\sin \pi/3$$

where $\rho = 1/\sin(\pi/3 + u)$.

The matrix (43) is the transfer matrix of an inhomogeneous 6-vertex model on the square lattice of width $n = n_0 + n_1$. In the next section we show that for the case where $n_0 = k$ and $n_1 = k + 1$ (sector $(k, k + 1)$ of the 6-vertex model on the lattice of width $2k + 1$) this transfer matrix has the special eigenvalue $\Lambda = 1$ independently of the values of the parameters u and u_j $(j = 1, \ldots, 2n + 1)$. Consequently, since $n = 2k + 1$, (44) reduces to

$$e^{-ik_j L} = 1 \qquad j = 1, \ldots, n \qquad (49)$$

and the associated energies of the quantum chain (13) are free-fermion-like.

4. The special eigenvalue

Applying the Bethe ansatz method to the auxiliary transfer matrix $\mathcal{T}(k) = \mathcal{T}(u)$ introduced in (43) one obtains the well-known NBAE [11, 12], which were considered in our paper [2]. We have shown there that these NBAE are consistent with the FFC (49) for the sectors $(n_0, n_1, n_2) = (k, k + 1, L - 2k - 1)$. However, it is more convenient here to follow an early paper of Baxter [15] who considered the most general integrable inhomogeneous 6-vertex model. Any eigenvalue $T(u)$ of the inhomogeneous model with $\rho = 1$ in (48) satisfies the equation

$$T(u)\, Q(u) = \left(\prod_{j=1}^{n} \sin(\pi/3 - u + u_j)\right) Q(u - 2\pi/3) + \left(\prod_{j=1}^{n} \sin(u - u_j)\right) Q(u + 2\pi/3)$$

$$(50)$$

where $Q(u)$ is an auxiliary trigonometric polynomial of degree n_0, namely,

$$Q(u) = \prod_{j=1}^{n_0} \sin(u - \alpha_j). \qquad (51)$$

It is clear from (50) that $T(u)$ is a trigonometric polynomial of degree n and from (48) the eigenvalues of (43) (where now $\rho \neq 1$) are given by

$$\Lambda(u) = T(u) \Big/ \prod_{j=1}^{n} \sin(\pi/3 + u - u_j). \qquad (52)$$

This last expression implies that we have an eigenvalue $\Lambda(u) = 1$ if

$$T(u) = \prod_{j=1}^{n} \sin(\pi/3 + u - u_j). \qquad (53)$$

The existence of this special eigenvalue was argued by Baxter for the more general case of the 8-vertex model with special values of the crossing parameters [16]. We can prove the existence of this eigenvalue for lattices with odd values of its width n. In this case we can rewrite Baxter's T–Q equation (50) as

$$f(u) + f(u + 2\pi/3) + f(u - 2\pi/3) = 0 \qquad (54)$$

where

$$f(u) \equiv Q(u + 2\pi/3) \prod_{j=1}^{n} \sin(u - u_j). \qquad (55)$$

It is now clear that $f(u)$ is a trigonometric polynomial of degree $n + n_0$. Equation (55) coincides with (6) in [4], where $f(u)$ is also a trigonometric polynomial.

In [4] it is shown that for any set of complex numbers u_j, $j = 1, \ldots, 2k + 1$, there exists a trigonometric polynomial $Z(u)$ of degree n on the variable u such that

$$f(u) \equiv Z(u) \prod_{j=1}^{2k+1} \sin(u - u_j) \tag{56}$$

satisfies $(54)^4$. The degree of f is equal to $n + n_0 = 3k + 1$ and the degree of Z is equal to the degree of Q, so that $k = n_0$. We obtain consequently that the construction of [4] corresponds to $n_0 = k$ and $n = n_0 + n_1 = 2k + 1$, i.e. $n_1 = k + 1$, which completes the proof of the existence of the special eigenvalue (53) for odd values of n.

5. The special wavefunction of the inhomogeneous 6-vertex model at $q^{2\mathrm{i}\pi/3}$

We consider several families of transfer matrices (43) corresponding to each distinct permutation $P = \{P_1, \ldots, P_n\}$. It follows from (41) that $A^{Q_1, \ldots, Q_n}_{P_1, \ldots, P_n}$ are the 2^n components $(Q_1, \ldots, Q_n = 0, 1)$ of an eigenvector of the transfer matrix (43). Now we are going to investigate these components for our special eigenvalue $\Lambda = 1$ using the generalization for an arbitrary number of particles n, i.e.,

$$\sum_{P=\{P_l, P_{l+1}\}} \left\{ \left(\sigma_{P_l, P_{l+1}} + q\, \mathrm{e}^{\mathrm{i}k_{P_{l+1}}}\right) A^{\ldots, 0, 1, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} - \mathrm{e}^{\mathrm{i}k_{P_{l+1}}} A^{\ldots, 1, 0, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} \right\} = 0$$

$$\sum_{P=\{P_l, P_{l+1}\}} \sigma_{P_l, P_{l+1}} A^{\ldots, Q, Q, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} = 0 \qquad Q = 1, 2 \tag{57}$$

$$\sum_{P=\{P_l, P_{l+1}\}} \left\{ \left(\sigma_{P_l, P_{l+1}} + q^{-1}\, \mathrm{e}^{\mathrm{i}k_{P_{l+1}}}\right) A^{\ldots, 1, 0, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} - \mathrm{e}^{\mathrm{i}k_{P_{l+1}}} A^{\ldots, 0, 1, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} \right\} = 0$$

where all indices shown by dots are fixed. When we constructed the S matrix in (33) we expressed $A^{Q_1, Q_2}_{P_1, P_2}$ as a linear combination of $A^{Q_1, Q_2}_{P_2, P_1}$ and $A^{Q_2, Q_1}_{P_2, P_1}$. Now, on the other hand, we intend to express $A^{Q_1, Q_2}_{P_1, P_2}$ as a linear combination of $A^{Q_2, Q_1}_{P_1, P_2}$ and $A^{Q_2, Q_1}_{P_2, P_1}$ ($Q_1 \neq Q_2$). Combining the first and the third equations of the set (27) we obtain

$$A^{1,0}_{P_1, P_2}(\mathrm{e}^{\mathrm{i}k_{P_2}} - \mathrm{e}^{\mathrm{i}k_{P_1}}) = (1 - q\, \mathrm{e}^{\mathrm{i}k_{P_1}})(1 - q^{-1}\mathrm{e}^{\mathrm{i}k_{P_2}}) A^{0,1}_{P_1, P_2} + \sigma_{P_2, P_1} A^{0,1}_{P_2, P_1}$$

$$A^{0,1}_{P_1, P_2}(\mathrm{e}^{\mathrm{i}k_{P_2}} - \mathrm{e}^{\mathrm{i}k_{P_1}}) = (1 - q^{-1}\, \mathrm{e}^{\mathrm{i}k_{P_1}})(1 - q\, \mathrm{e}^{\mathrm{i}k_{P_2}}) A^{1,0}_{P_1, P_2} + \sigma_{P_2, P_1} A^{1,0}_{P_2, P_1}. \tag{58}$$

Changing the variables as in (45) these equations are replaced by

$$A^{1,0}_{P_1, P_2} \sin(u_{P_2} - u_{P_1}) = \frac{\sqrt{3}}{2} \mathrm{e}^{\mathrm{i}(u_{P_2} - u_{P_1})} A^{0,1}_{P_1, P_2} + \sin(u_{P_1} - u_{P_2} + \pi/3) A^{0,1}_{P_2, P_1}$$

$$A^{0,1}_{P_1, P_2} \sin(u_{P_2} - u_{P_1}) = \frac{\sqrt{3}}{2} \mathrm{e}^{\mathrm{i}(u_{P_1} - u_{P_2})} A^{1,0}_{P_1, P_2} + \sin(u_{P_1} - u_{P_2} + \pi/3) A^{1,0}_{P_2, P_1}. \tag{59}$$

Generalizing these equations for arbitrary n and considering $\tilde{A}$ instead of A (as in (47)) we obtain

$$\tilde{A}^{\ldots, 1, 0, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} \sin\left(u_{P_{l+1}} - u_{P_l}\right) = \frac{\sqrt{3}}{2} \tilde{A}^{\ldots, 0, 1, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} + \sin\left(u_{P_l} - u_{P_{l+1}} + \pi/3\right) \tilde{A}^{\ldots, 0, 1, \ldots}_{\ldots, P_{l+1}, P_l, \ldots} = 0$$

$$\tilde{A}^{\ldots, 0, 1, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} \sin\left(u_{P_{l+1}} - u_{P_l}\right) = \frac{\sqrt{3}}{2} \tilde{A}^{\ldots, 1, 0, \ldots}_{\ldots, P_l, P_{l+1}, \ldots} + \sin\left(u_{P_l} - u_{P_{l+1}} + \pi/3\right) \tilde{A}^{\ldots, 1, 0, \ldots}_{\ldots, P_{l+1}, P_l, \ldots} = 0. \tag{60}$$

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[4] $Z(u)$ is the partition function of the inhomogeneous 6-vertex model with domain wall boundary conditions and with rapidities $\{u, u_j\}$, $j = 1, \ldots, 2k + 1$.

We also have a generalization of the second equation in (27)

$$\sin\left(u_{P_{l+1}} - u_{P_l} + \pi/3\right) \tilde{A}^{\dots,Q,Q,\dots}_{\dots,P_l,P_{l+1},\dots} = -\sin\left(u_{P_l} - u_{P_{l+1}} + \pi/3\right) \tilde{A}^{\dots,Q,Q,\dots}_{\dots,P_{l+1},P_l,\dots} \qquad (Q = 0, 1).$$

(61)

Moreover, due to FFC (49) equation (40) leads to the cyclic symmetry

$$A^{Q_1,\dots,Q_n}_{P_1,\dots,P_n} = A^{Q_2,\dots,Q_n,Q_1}_{P_2,\dots,P_n,P_1}$$

(62)

which is also valid for $\tilde{A}$.

Let us begin with the simplest nontrivial case $k = 1$, where the sector of appearance of free-fermion levels is $(1, 2, L - 3)$. Equation (17) for the eigenvectors can be written as

$$|n_0, n_1\rangle = |1, 2\rangle = \sum_{\{Q\}} \sum_{1 \leqslant x_1, x_2, x_3 \leqslant L} f(x_1, Q_1; x_2, Q_2; x_3, Q_3)|x_1, Q_1; x_2, Q_2; x_3, Q_3\rangle \quad (63)$$

where we sum over the three sequences of $\{Q\}$: $\{0, 1, 1\}$, $\{1, 0, 1\}$ and $\{1, 1, 0\}$, which correspond to the configurations where $n_0 = 1$, $n_1 = 2$ ($n = n_0 + n_1 = 3$). The amplitudes are given by the ansatz (36)

$$f(x_1, Q_1; x_2, Q_2; x_3, Q_3) = \sum_P A^{Q_1,Q_2,Q_3}_{P_1,P_2,P_3} e^{i(k_{P_1}x_1 + k_{P_2}x_2 + k_{P_3}x_3)}.$$

(64)

In the above expression there are three types of parameters $\{A\}$ which are related among themselves by the cyclic symmetry:

$$\tilde{A}^{0,1,1}_{P_1,P_2,P_3} = \tilde{A}^{1,1,0}_{P_2,P_3,P_1} = \tilde{A}^{1,0,1}_{P_3,P_1,P_2}.$$

(65)

Before proceeding let us introduce the simplified notation

$$s_{P_1,P_2} = \frac{\sin\left(u_{P_1} - u_{P_2} + \pi/3\right)}{\sin(\pi/3)}$$

$$d_{P_1,P_2} = \frac{\sin\left(u_{P_1} - u_{P_2}\right)}{\sin(\pi/3)}.$$

(66)

Relation (61) can be reduced to the equation

$$s_{P_3,P_2} \tilde{A}^{0,1,1}_{P_1,P_2,P_3} = -s_{P_2,P_3} \tilde{A}^{0,1,1}_{P_1,P_3,P_2}$$

(67)

so that due to (65)

$$\tilde{A}^{0,1,1}_{P_1,P_2,P_3} = \pm C\{P_1\}s_{P_2,P_3}$$

(68)

where the sign depends on the parity of the permutation $P = \{P_1, P_2, P_3\}$ and $C\{i\}$, $i = 1, 2, 3$, are unknown coefficients.

Relations (60) give us in particular the equation

$$\tilde{A}^{1,0,1}_{P_1,P_2,P_3}d_{P_2,P_1} = \tilde{A}^{0,1,1}_{P_1,P_2,P_3} + s_{P_1,P_2}\tilde{A}^{0,1,1}_{P_2,P_1,P_3}.$$

(69)

Using the cyclic symmetries (65) and (68) we obtain

$$C\{P_2\}s_{P_3,P_1}d_{P_2,P_1} = C\{P_1\}s_{P_2,P_3} - C\{P_2\}s_{P_1,P_2}s_{P_1,P_3}$$

(70)

which together with the identity

$$s_{P_3,P_1}d_{P_2,P_1} + s_{P_1,P_2}s_{P_1,P_3} = s_{P_2,P_3}$$

(71)

gives us $C\{P_2\} = C\{P_2\} = C$, i.e., up to a normalization factor we have

$$\tilde{A}^{0,1,1}_{P_1,P_2,P_3} = \pm s_{P_2,P_3}$$

(72)

where the sign depends on the parity of the permutation P.

Consider further the next sector $(2, 3, L - 5)$, i.e., $n_0 = 2, n_1 = 3$ and $n = n_0 + n_1 = 5$. In this case we have two sets of $\{\tilde{A}\}$, which are related due to the cyclic symmetry, namely,

$$\tilde{A}^{0,0,1,1,1}_{P_1,P_2,P_3,P_4,P_5} = \tilde{A}^{0,1,1,1,0}_{P_2,P_3,P_4,P_5,P_1} = \cdots = \tilde{A}^{1,0,0,1,1}_{P_5,P_1,P_2,P_3,P_4}$$

$$\tilde{A}^{0,1,0,1,1}_{P_1,P_2,P_3,P_4,P_5} = \tilde{A}^{1,0,1,1,0}_{P_2,P_3,P_4,P_5,P_1} = \cdots = \tilde{A}^{1,0,1,0,1}_{P_5,P_1,P_2,P_3,P_4}. \tag{73}$$

Let us begin with the first set. Relation (61) is solved by the ansatz

$$\tilde{A}^{0,0,1,1,1}_{P_1,P_2,P_3,P_4,P_5} = \pm C\{P_1, P_2\} s_{P_1,P_2} s_{P_3,P_4} s_{P_3,P_5} s_{P_4,P_5} \tag{74}$$

where the sign depends on the parity of the permutation $P = \{P_1, P_2, P_3, P_4, P_5\}$ and $C\{i, j\}, i, j = 1, 2, 3, 4, 5$, are symmetric unknown coefficients $C\{i, j\} = C\{j, i\}$. The system (60) contains in particular the following three equations:

$$\tilde{A}^{0,1,0,1,1}_{P_1,P_2,P_3,P_4,P_5} d_{P_3,P_2} = \tilde{A}^{0,0,1,1,1}_{P_1,P_2,P_3,P_4,P_5} + s_{P_2,P_3} A^{0,0,1,1,1}_{P_1,P_3,P_2,P_4,P_5}$$

$$\tilde{A}^{0,1,0,1,1}_{P_2,P_1,P_3,P_4,P_5} d_{P_3,P_1} = \tilde{A}^{0,0,1,1,1}_{P_2,P_1,P_3,P_4,P_5} + s_{P_1,P_3} A^{0,0,1,1,1}_{P_2,P_3,P_1,P_4,P_5} \tag{75}$$

$$\tilde{A}^{1,0,0,1,1}_{P_1,P_2,P_3,P_4,P_5} d_{P_2,P_1} = \tilde{A}^{0,1,0,1,1}_{P_1,P_2,P_3,P_4,P_5} + s_{P_1,P_2} \tilde{A}^{0,1,0,1,1}_{P_2,P_1,P_3,P_4,P_5}.$$

Excluding from this system $\tilde{A}^{0,1,0,1,1}_{\cdots}$, using the cyclic symmetry

$$\tilde{A}^{1,0,0,1,1}_{P_1,P_2,P_3,P_4,P_5} = \tilde{A}^{0,0,1,1,1}_{P_2,P_3,P_4,P_5,P_1}$$

and limiting ourselves to the unit permutation, we get

$$\tilde{A}^{0,0,1,1,1}_{2,3,4,5,1} d_{2,1} d_{3,2} d_{3,1} = \left(\tilde{A}^{0,0,1,1,1}_{1,2,3,4,5} + s_{2,3}\tilde{A}^{0,0,1,1,1}_{1,3,2,4,5}\right) d_{3,1} + \left(\tilde{A}^{0,0,1,1,1}_{2,1,3,4,5} + s_{1,3}\tilde{A}^{0,0,1,1,1}_{2,3,1,4,5}\right) d_{3,2} s_{1,2}. \tag{76}$$

Inserting here the ansatz (74) and using the antisymmetry of $d_{i,j}$, we obtain

$$C\{1, 2\}s_{1,2}s_{3,4}s_{3,5}(d_{2,3}s_{2,1} - d_{1,3}) + C\{1, 3\}d_{1,3}s_{1,3}s_{2,3}s_{2,4}s_{2,5}$$
$$+ C\{2, 3\}d_{2,3}s_{2,3}(d_{1,2}d_{1,3}s_{4,1}s_{5,1} - s_{1,2}s_{1,3}s_{1,4}s_{1,5}) = 0. \tag{77}$$

Using the identity

$$d_{2,3}s_{2,1} - d_{1,3} = -d_{1,2}s_{2,3} \tag{78}$$

and removing a common multiplier $s_{2,3}$ from (77), we obtain the more simple equation

$$-C\{1, 2\}d_{1,2}s_{1,2}s_{3,4}s_{3,5} + C\{1, 3\}d_{1,3}s_{1,3}s_{2,4}s_{2,5}$$
$$+ C\{2, 3\}d_{2,3}(d_{1,2}d_{1,3}s_{4,1}s_{5,1} - s_{1,2}s_{1,3}s_{1,4}s_{1,5}) = 0. \tag{79}$$

By interchanging indices 1 and 2 we obtain a distinct equation. If we now exclude $C\{1, 2\}$ from these two equations we get a relation between $C\{2, 3\}$ and $C\{1, 3\}$, namely,

$$C\{2, 3\}d_{2,3}(d_{1,2}d_{1,3}s_{2,1}s_{4,1}s_{5,1} + s_{1,2}s_{1,4}s_{1,5}(s_{2,3} - s_{2,1}s_{1,3}))$$
$$= C\{1, 3\}d_{1,3}(d_{1,2}d_{2,3}s_{1,2}s_{4,2}s_{5,2} + s_{2,1}s_{2,4}s_{2,5}(s_{1,2}s_{2,3} - s_{1,3})). \tag{80}$$

This last relation takes a nice form if we use the identities

$$s_{2,3} - s_{2,1}s_{1,3} = d_{1,2}d_{1,3} \qquad s_{1,2}s_{2,3} - s_{1,3} = d_{1,2}d_{2,3} \tag{81}$$

and remove the common factors $d_{1,2}$, $d_{1,3}$ and $d_{2,3}$, i.e.,

$$C\{2, 3\}(s_{2,1}s_{4,1}s_{5,1} + s_{1,2}s_{1,4}s_{1,5}) = C\{1, 3\}(s_{1,2}s_{4,2}s_{5,2} + s_{2,1}s_{2,4}s_{2,5}). \tag{82}$$

Using standard trigonometric identities one can show that the left-hand combination

$$s_{2,1}s_{4,1}s_{5,1} + s_{1,2}s_{1,4}s_{1,5} = \tfrac{2}{3}\{\cos(u_1 + u_2 - u_4 - u_5)$$
$$+ \cos(u_1 - u_2 + u_4 - u_5) + \cos(u_1 - u_2 - u_4 + u_5)\} \tag{83}$$

has an S_4 symmetry, and consequently (82) reduces to

$$C\{2, 3\} = C\{1, 3\}. \tag{84}$$

This means that C does not depend on its indices and we have up to a normalization factor

$$\tilde{A}^{0,0,1,1,1}_{P_1,P_2,P_3,P_4,P_5} = \pm s_{P_1,P_2} s_{P_3,P_4} s_{P_3,P_5} s_{P_4,P_5}. \tag{85}$$

The second set of amplitudes can be found from the first relation in (75). For example,

$$\tilde{A}^{0,1,0,1,1}_{1,2,3,4,5} = d^{-1}_{2,3} s_{4,5}(s_{1,3} s_{2,3} s_{2,4} s_{2,5} - s_{1,2} s_{3,4} s_{3,5})$$
$$\equiv d^{-1}_{2,3} s_{4,5}\{s_{2,3} s_{2,5}(s_{1,3} s_{2,4} - s_{1,2} s_{3,4}) + s_{1,2} s_{3,4}(s_{2,3} s_{2,5} - s_{3,5})\} \tag{86}$$

and using the identities

$$s_{1,3} s_{2,4} - s_{1,2} s_{3,4} = d_{2,3} s_{4,1} \qquad s_{2,3} s_{2,5} - s_{3,5} = d_{2,3} s_{5,2} \tag{87}$$

we obtain

$$\tilde{A}^{0,1,0,1,1}_{1,2,3,4,5} = s_{4,5}(s_{2,3} s_{2,5} s_{4,1} + s_{1,2} s_{3,4} s_{5,2}), \tag{88}$$

or equivalently by using some additional identities we get

$$\tilde{A}^{0,1,0,1,1}_{1,2,3,4,5} = s_{4,5}(s_{2,3} s_{2,4} s_{5,1} + s_{1,2} s_{3,5} s_{4,2}). \tag{89}$$

In a similar way we can derive the general answer

$$\tilde{A}^{0,1,0,1,1}_{P_1,P_2,P_3,P_4,P_5} \Big/ s_{P_4,P_5} = s_{P_2,P_3} s_{P_2,P_5} s_{P_4,P_1} + s_{P_1,P_2} s_{P_3,P_4} s_{P_5,P_2}$$
$$= s_{P_2,P_3} s_{P_2,P_4} s_{P_5,P_1} + s_{P_1,P_2} s_{P_3,P_5} s_{P_4,P_2}. \tag{90}$$

Equations (72) and (85) induce us to conjecture that for an arbitrary n the amplitudes $\{\tilde{A}\}$ of the special wavefunction of the inhomogeneous 6-vertex model are given by the ansatz

$$A^{0,\dots,0,1,\dots,1}_{P_1,\dots,P_k,P_{k+1},\dots,P_{2k+1}} = \prod_{1 \leqslant i < j \leqslant k} s_{P_i,P_j} \prod_{k+1 \leqslant i < j \leqslant 2k+1} s_{P_i,P_j} \tag{91}$$

where k and $k + 1$ are the numbers of particles of species 0 and 1, respectively. We checked this formula for $n = 7$ analytically and for $n = 9$ using a brute-force diagonalization. It is a challenge to prove the validity of this formula for an arbitrary odd number n. The remaining amplitudes can be found by using this ansatz and successive application of the 'recursion' relation (60). This completes our discussion of the $SU(2)$ periodic case.

6. Summary and conclusions

In previous papers [1, 2] the existence of free-fermion-like energies for the anisotropic $SU(N)$ Perk–Schultz model with anisotropy parameter $q = \exp i\pi(N - 1)/N$ was shown. These solutions were found for general values of N in the case of a free boundary condition, where the model is $SU_q(N)$ invariant, and for the $SU(3)$ case in the periodic case.

In section 2 we have shown that the above observations, for the case of free boundaries, could be explained by a conjecture stating the existence of a special factorizable eigenvalue of the auxiliary inhomogeneous transfer matrix of an $SU_q(N - 1)$ vertex model with the same value of the anisotropy. Although we believe that a general derivation of such a factorizable eigenvalue would be possible we restricted our analytical work to the simplest case of the periodic $SU(3)$ Perk–Schultz model at $q = \exp i\frac{2\pi}{3}$. In this case the associated transfer matrix is that of the inhomogeneous 6-vertex model and the existence of the factorizable eigenvalue, as shown in section 4, follows from the T–Q Baxter equation (50).

In section 3 we have reviewed the coordinate Bethe ansatz and shown how to relate the wavefunction components of the eigenvectors of the $SU(3)$ quantum chain with periodic boundaries in terms of the components of the eigenvectors of the inhomogeneous 6-vertex model. In particular, all the free-fermion-like energies are related to a single factorizable eigenvalue of the inhomogeneous 6-vertex model.

In section 5, exploring the existence of the free-fermion-like solutions of the $SU(3)$ chain at $q = \exp \mathrm{i} \frac{2\pi}{3}$, we have shown how to produce the recurrence relations that allow the computation of the wave vector amplitudes related to the special factorizable eigenvector of the 6-vertex model. These relations, although not simple, give us a systematic way to derive all the eigenfunctions of the free-fermion part of the eigenspectrum of the periodic $SU(3)$ Perk–Schultz model at $q = \exp \mathrm{i} \frac{2\pi}{3}$.

As an application let us consider the free-fermion branch of this last model in the sector $S_k = (k, k+1, L-2k-1)$. From (14) the corresponding eigenenergies are given by

$$E_I = -\sum_{j \in I}(1 + 2\cos(2\pi j/L))$$

where I is any subset of $(1, 2, \ldots, L)$ with $n = 2k + 1$ distinct elements. Enumerating these elements by the index $\alpha = 1, 2, \ldots, n$, the wavefunction is given by

$$f(x_1, Q_1; \ldots; x_n, Q_n) = \sum_P A^{Q_1,\ldots,Q_n}_{P_1,\ldots,P_n} \mathrm{e}^{\mathrm{i}(k_{P_1}x_1 + \cdots + k_{P_n}x_n)}$$

where $k_\alpha = 2\pi j_\alpha/L, \alpha = (1, 2, \ldots, n)$ are the momenta of the elementary free-fermion excitations.

Let us limit ourselves to the subsets I with elements j_α, satisfying the constraint $j < m$ or $L - m < j$, where m is a positive integer. Due to conjecture 2 of [2] the lowest eigenenergy in the sector $S_k = (k, k+1, L-2k-1)$ belongs to this part of spectrum if we choose $m > k$.

Now we fix k and m with $m > k$ and consider the bulk limit $L \to \infty$. Due to the above-mentioned constraints for the values of k we have two possibilities: $k_\alpha \to 0$ or $k_\alpha \to 2\pi$. Consequently, from (46) all the parameters u_α that fix the auxiliary 6-vertex model become equal to $\pi/2$ and we obtain the homogeneous model with the special eigenvector $A^{Q_1,\ldots,Q_n}$ which (up to a sign factor) does not depend on the particular permutation P! The wavefunction can then be written as

$$f(y_1, Q_1; \ldots; y_n, Q_n) = A^{Q_1,\ldots,Q_n} \sum_P (-1)^P \mathrm{e}^{2\pi \mathrm{i}(j_{P_1}y_1 + \cdots + j_{P_n}y_n)} \tag{92}$$

where we introduced the new coordinates $y_i = x_i/L \ (i = 1, \ldots, n)$. In particular, this result shows that in the sectors $(k, k+1, \infty)$ there exist a lot of eigenstates (including the one with lowest eigenenergy in the sector) whose wavefunction components are given by the product of Slater determinants and the components of the ground state wavefunction of the XXZ model with $2k + 1$ sites. Let us consider, for example, the sector $S_2 = (2, 3, L - 5)$. In order to obtain the lowest eigenenergy in this sector we chose $I = 1, 2; L - 2, L - 1, L$. The Slater determinant in (92) reduces to the Vandermonde determinant and we obtain (up to a normalization)

$$f(y_1, Q_1; \ldots; y_5, Q_5) = \prod_{1 \leqslant j < k \leqslant 5} \sin \pi(y_j - y_k) A^{Q_1,\ldots,Q_5} \tag{93}$$

where we have

$$A^{00111} = A^{01110} = A^{11100} = A^{11001} = A^{10011} = 1$$
$$A^{01011} = A^{10110} = A^{01101} = A^{11010} = A^{10101} = 2.$$

From these components we see, for example, that the probability of finding 0-particles, separated by 1-particles, is equal to $2 * 2/(1 * 1 + 2 * 2) = 4/5$.

Moreover, it is important to mention that when the inhomogeneity of the auxiliary 6-vertex model vanishes, the wavefunction that corresponds to the special factorizable eigenvalue coincides with the ground state wavefunction of the XXZ spin Hamiltonian with anisotropy $\Delta = -1/2$. For odd values of lattice sizes the amplitudes of this last ground state wavefunction show quite interesting combinatorial properties.

Due to the conjecture announced at the end of section 2, we have a generalization of this special wavefunction to the $SU(N-1)$ invariant case ($q = e^{i(N-1)\pi/N}$). So we suspect that the ground state function of the $SU(N-1)$ invariant spin chain can also exhibit interesting combinatorial properties for the special value $q = e^{i(N-1)\pi/N}$. Indeed, using a bruteforce diagonalization of these quantum invariant chains we have found that the ratio defined by

$$R_L = \frac{\left(\sum_i v_i\right)^2}{\sum_i v_i^2} \tag{94}$$

where $\{v_i\}$ are the wavefunction components of the ground state, has a simple form depending on the boundary condition and on the parity of the lattice size. For the L-site XXZ spin chain at $\Delta = -1/2$ we have $R_L = \sqrt{3}^{\alpha}$ [7], where $\alpha = L - 1$ or $\alpha = L$ depending on whether the length L of the chain is odd or even, respectively. In the case where L is odd the chain has a boundary condition periodic or $SU_q(2)$ invariant, and for even values of L the chain has a boundary condition of twisted type or an $SU_q(2)$ invariant one. We can present these results in a compact form:

$$R_{L+2}/R_L = 3 \qquad R_1 = 1 \quad R_2 = 3. \tag{95}$$

The numerical results coming from brute-force numerical diagonalizations of chains with $SU_q(N)$ symmetry (free boundary condition) followed by a fitting with special irrational numbers give us the values of R_L for the next cases:

- $SU_q(3)$ with $q = -e^{i\pi/4}$ ($2 \leqslant L \leqslant 9$)

$$R_{L+3}/R_L = (1 + \sqrt{2})^3$$
$$R_1 = 1 \qquad R_2 = 1 + \sqrt{2} \qquad R_3 = (1 + \sqrt{2})^3. \tag{96}$$

- $SU_q(4)$ with $q = -e^{i\pi/5}$ ($2 \leqslant L \leqslant 8$)

$$R_{L+4}/R_L = (5 + 2\sqrt{5})^2$$
$$R_1 = 1 \qquad R_2 = \sqrt{5} \qquad R_3 = 5 + 2\sqrt{5} \qquad R_4 = (5 + 2\sqrt{5})^2. \tag{97}$$

- $SU_q(5)$ and $q = -e^{i\pi/6}$ ($2 \leqslant L \leqslant 7$)

$$R_{L+5}/R_L = (2 + \sqrt{3})^5$$
$$R_1 = 1 \qquad R_2 = (2 + \sqrt{3})/\sqrt{3} \qquad R_3 = (2 + \sqrt{3})^2/\sqrt{3} \tag{98}$$
$$R_4 = (2 + \sqrt{3})^3 \qquad R_5 = (2 + \sqrt{3})^5.$$

The numbers R_L, $L < N$, which are necessary for the use of the recursion relations can be found from the explicit expression of the corresponding wavefunction. Using an approach described in appendix B of [2] one can show that for $L < N$

$$R_L = \left(\frac{1+x}{1-x}\right)^{L-1} \prod_{k=2}^{L} \frac{1-x^k}{1+x^k} \qquad x = -q = e^{i\pi/N}. \tag{99}$$

For the cases where $L \geqslant N$ although we cannot prove, our numerical results indicate the nice recursion relation

$$R_{L+N-1} = R_{N-1} R_L. \tag{100}$$

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